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Instability of vibrations of a mass that moves uniformly along a beam on a periodically inhomogeneous foundation

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Abstract

The stability of vibrations of a mass that moves uniformly along an Euler–Bernoulli beam on a periodically inhomogeneous continuous foundation is studied. The inhomogeneity of the foundation is caused by a slight periodical variation of the foundation stiffness. The moving mass and the beam are assumed to be always in contact. With the help of a perturbation analysis it is shown analytically that vibrations of the system may become unstable. The physical phenomenon that lies behind this instability is parametric resonance that occurs because of the periodic (in time) variation of the foundation stiffness under the moving mass. The first instability zone is found in the system parameters within the first approximation of the perturbation theory. The location of the zone is strongly dependent on the spatial period of the inhomogeneity and on the weight of the moving mass. The larger this period is and/or the smaller the mass, the higher the velocity is at which the instability occurs.

1. Introduction

Vibrations of a vehicle moving over a flexible guideway can become unstable. This instability arises due to the interaction between the vehicle and the guideway and shows itself in the exponential increase of the amplitude of the vibrations in time. The energy needed for this increase is supplied by an external source (an engine) that maintains the motion of the vehicle.

If the guideway is modelled as a (long) structure that is homogeneous in the direction of motion, then the instability can only occur if the vehicle moves with a velocity that exceeds the minimum phase velocity of waves in the structure [1–8]. In real structures, like a railway track, the

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latter is very high, unless the rails are significantly stressed in the axial direction due to the temperature extension [4,9,10] or the track is built on a soft soil [6].

Most real guideways are however, periodically inhomogeneous in the longitudinal direction. For example, a conventional railway track is inhomogeneous because of the sleepers and/or corrugation of the rail surfaces. A catenary system (for trains or trams) is inhomogeneous due to the hangers that support the contact wire. A guideway for magnetically levitated trains, being a multi-span structure, is also a highly inhomogeneous system, etc. There are an impressive number of papers that deal with dynamics of periodically inhomogeneous guideways, see, for example [11–17]. To our knowledge, however, only two of these papers [18,19] are devoted to the instability of a moving vehicle that can arise due to the periodicity of the guideway. In Chung and Genin [18], the stability of a two-mass oscillator was studied as it moved uniformly on a multi-span beam. It was shown that the system can loose its stability because of the parametric resonance that is caused by periodical variation of the guideway stiffness under the moving vehicle. The same conclusion was drawn in Ref. [19] where the stability of a moving mass on a string supported by a distributed, periodically inhomogeneous foundation was investigated.

In this paper the stability of a mass is considered as it moves with a constant speed on a beam that is supported by a periodically inhomogeneous visco-elastic foundation. It is assumed that the stiffness of this foundation varies slightly about its mean value. With the help of a perturbation technique, it is shown analytically that vibrations of the system can become unstable as the mass moves with certain velocities that are substantially smaller than the minimum phase velocity of waves in the respective homogeneous structure.

2. The model and the governing equations

Fig. 1 shows the model under consideration, which is composed of a moving mass and an Euler–Bernoulli beam on a visco-elastic foundation. The mass moves along the beam uniformly, with a constant velocity V and remains always in contact with the beam. The stiffness of the foundation varies periodically along the beam and is defined by the following expression:

$$k(x) = k_f(1 + \mu\cos(\chi x)), \quad \chi = 2\pi/d, \tag{1}$$

with k_f the mean stiffness of the foundation, d the period of the inhomogeneity, χ the wave number of the inhomogeneity and $\mu \ll 1$ a dimensionless small parameter.

The governing equations that describe small vertical vibrations of the system are

$$\begin{aligned}
&\rho A_{cs}\frac{\partial^2 w}{\partial t^2} + EI\frac{\partial^4 w}{\partial x^4} + \mu v_f\frac{\partial w}{\partial t} + k(x)w = 0,\\
&[w]_{x=Vt} = \left[\frac{\partial w}{\partial x}\right]_{x=Vt} = \left[\frac{\partial^2 w}{\partial x^2}\right]_{x=Vt} = 0,\\
&w|_{x=Vt} = u_0,\\
&EI\left[\frac{\partial^3 w}{\partial x^3}\right]_{x=Vt} = -m\frac{\mathrm{d}^2 w_0}{\mathrm{d}t^2} - mg
\end{aligned} \tag{2}$$

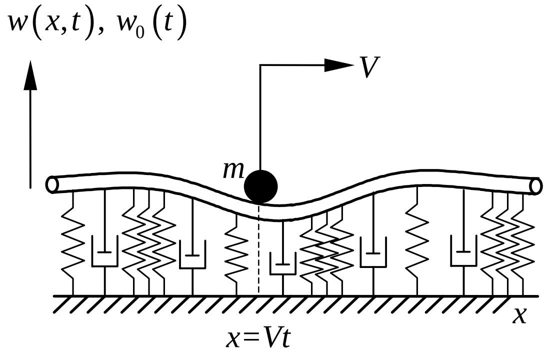

Fig. 1. Uniform motion of a mass along a beam on a periodically inhomogeneous foundation.

with $w(x,t)$ and $w_0(t)$ the vertical deflections of the beam and the mass relative to their equilibrium positions, E and ρ are Young's modulus and the mass density of the beam's material, A_{cs} the cross-sectional area of the beam, I the moment of inertia of the beam's cross-section, μv_f the small viscosity of the foundation, $\delta(\ldots)$ the Dirac delta function and where the square brackets indicate the difference between the bracketed quantities on either side of the limit $x = Vt$, for example $[w]_{x=Vt} = w(x = Vt + 0, t) - w(x = Vt - 0, t)$.

The first equation of the system (2) gives the dynamic balance of forces acting on a differential element of the beam. Equations $[x]_{x=Vt} = 0$ and $[\partial w/\partial x]_{x=Vt} = 0$ ensure that the deflection of the beam and its slope are continuous in the contact point. Equation $[\partial^2 w/\partial x^2]_{x=Vt} = 0$ implies that there is no external moment applied at the contact point. Equation $w|_{x=Vt} = u_0$ is the continuity condition that implies that the mass and the beam are always in contact.

The last equation of the system is the balance of vertical forces that act on the moving mass. Note that in this equation the dead weight of the mass mg can be omitted in the further investigation. This can be done deliberately since this weight (external constant force) may not influence the system stability. In the simplest case of a homogeneous foundation $k(x) = k_f = const$ there is a critical velocity $V_{cr} = \sqrt{4k_f EI/m^2}$ for which the beam displacement grows *linearly* in time. Usually, in the case of a periodically inhomogeneous foundation such an effect occurs if the normal frequency of the mass equals the frequency of inhomogeneity in the contact point. Because of these reasons the term mg is omitted from the right-hand part of the last equation in Eq. (2) since the interest is to determine whether the beam displacement grows in time *exponentially*.

Since the inhomogeneity of the beam's foundation is small, a perturbation technique [20] can be applied to analyze the system of Eq. (2). The basic idea of the technique that will be first applied is that the presence of a small inhomogeneity cannot significantly influence the solution to the problem. Therefore, this solution can be sought in the following form:

$$w(x,t) = w^{(0)}(x,t) + \mu w^{(1)}(x,t) + \cdots, \quad w_0(t) = w_0^{(0)}(t) + \mu w_0^{(1)}(t) + \cdots, \tag{3}$$

where $w^{(0)}(x,t)$ and $w_0^{(0)}(t)$ are solutions to the unperturbed problem, e.g., to the system of Eq. (2) in which the small parameter μ is set to zero. Physically, these solutions describe vibrations of the moving mass on a beam that is supported by a homogeneous visco-elastic foundation with the stiffness k_f [1,4]. Obviously, the governing equations for the unperturbed problem are

$$\begin{aligned} &\rho A_{cs}\frac{\partial^2 w^{(0)}}{\partial t^2}+EI\frac{\partial^4 w^{(0)}}{\partial x^4}+k_f w^{(0)}=0,\\ &\left[w^{(0)}\right]_{x=Vt}=\left[\frac{\partial w^{(0)}}{\partial x}\right]_{x=Vt}=\left[\frac{\partial^2 w^{(0)}}{\partial x^2}\right]_{x=Vt}=0,\\ &w^{(0)}\big|_{x=Vt}=w_0^{(0)},\\ &EI\left[\frac{\partial^3 w^{(0)}}{\partial x^3}\right]_{x=Vt}=-m\frac{\mathrm{d}^2 w_0^{(0)}}{\mathrm{d}t^2}. \end{aligned} \tag{4}$$

The second terms $\mu w^{(1)}(x,t)$ and $\mu w_0^{(1)}(t)$ in expressions (3) should be much smaller than $w^{(0)}(x,t)$ and $w_0^{(0)}(t)$, respectively. To find the system of equations for the variables $w^{(1)}(x,t)$ and $w_0^{(1)}(t)$, expressions (3) have to be substituted into the system of Eq. (2) after which all terms that are proportional to μ should be gathered. With the use of expression (1) this yields

$$\begin{aligned} &\rho A_{cs}\frac{\partial^2 w^{(1)}}{\partial t^2}+EI\frac{\partial^4 w^{(1)}}{\partial x^4}+k_f w^{(1)}=-k_f w^{(0)}\cos(\chi x)-v_f\frac{\partial^2 w^{(0)}}{\partial t},\\ &\left[w^{(1)}\right]_{x=Vt}=\left[\frac{\partial w^{(1)}}{\partial x}\right]_{x=Vt}=\left[\frac{\partial^2 w^{(1)}}{\partial x^2}\right]_{x=Vt}=0,\\ &w^{(1)}\big|_{x=Vt}=u_0^{(1)},\\ &EI\left[\frac{\partial^3 w^{(1)}}{\partial x^3}\right]_{x=Vt}=-m\frac{\mathrm{d}^2 w_0^{(1)}}{\mathrm{d}t^2}, \end{aligned} \tag{5}$$

Thus, to study the original problem (2), first, the unperturbed problem (4) should be solved. Second, the unperturbed solutions $w^{(0)}(x,t)$ and $w_0^{(0)}(t)$ have to be substituted into the system of Eq. (5), in which these solutions coupled with the inhomogeneity will serve as an excitation. In the next Section the first step of this analysis is accomplished, e.g., the unperturbed problem is studied.

3. Solution to the unperturbed problem

The system of Eq. (4) that describes vibrations of the moving mass on a beam that is supported by a homogeneous visco-elastic foundation has been studied in Refs. [1,4]. As shown in Ref. [4], the characteristic equation that defines the natural frequencies of vibrations of the mass on the beam is given as

$$\frac{m\Omega^2}{EI}+\left(4\mathrm{i}\sum_n\frac{(k-k_n)}{(k-k_1)(k-k_2)(k-k_3)(k-k_4)}\bigg|_{k=k_n}\right)^{-1}=0, \tag{6}$$

where $\mathrm{i}=\sqrt{-1}$, Ω the natural frequency and k_n, $n=1,\dots,4$ the roots of the equation

$$-\rho A_{cs}(\Omega-kV)^2+EIk^4+k_f+\mathrm{i}v_f(\Omega-kV)=0, \tag{7}$$

that have a positive imaginary part and calculated with $v_f \to 0$. In this limit, which implies a transition to a system with infinitely small damping, Eq. (7) reduces to the following dispersion equation for the beam on the elastic foundation (in the reference system that moves together with the mass):

$$-\rho A_{cs}(\Omega - kV)^2 + EIk^4 + k_f = 0. \tag{8}$$

By using the technique presented in Refs. [1,4], it can be shown that if the velocity of the mass is smaller than the minimum phase velocity of waves in the beam on the inhomogeneous foundation, e.g., if $V < V_{ph}^{min} = (4k_f EI/\rho^2 A_{cs}^2)^{1/4}$, then the roots of the characteristic Eq. (6) are real. This implies that the heave vibrations of the moving mass on the beam are harmonic in this case.

Obviously, the mass can vibrate harmonically if and only if these vibrations do not perturb waves in the beam (otherwise, the radiation damping would cause decay of the vibrations). Mathematically, this implies that the wave numbers that are found as the roots of the dispersion Eq. (8) may not be real. The system of inequalities that does not permit the roots of Eq. (8) to be real can be found analytically to give

$$\Omega < \Omega^{cut\text{-}off} = \sqrt{k_f/\rho A_{cs}},$$
$$V < V^{cr}(\Omega),$$
$$V^{cr}(\Omega) = \left(\frac{EI(-\rho^2 A_{cs}^2 \Omega^4 + 20\rho A_{cs} k_f \Omega^2 + 8k_f^2 - \Omega\sqrt{\rho A_{cs}}(\rho A_{cs}\Omega^2 + 8k_f)^{3/2})}{2k_f \rho^2 A_{cs}^2}\right)^{1/4}. \tag{9}$$

In accordance with system (9) the frequency of harmonic vibrations of the mass cannot be larger that a certain critical value $\Omega^*(V)$ that depends on the velocity of the mass. This dependence, which is normally referred to as a bifurcation curve, is depicted in Fig. 2 by the solid line. This

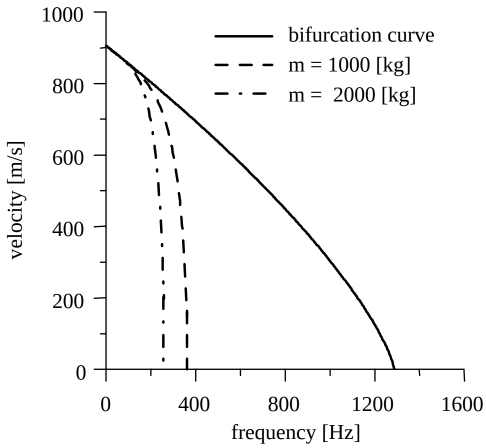

Fig. 2. Bifurcation curve and the natural frequency of the mass versus velocity for two different magnitudes of the mass: —, bifurcation curve, – – – $m = 1000$ kg; – · –, $m = 2000$ kg.

figure was drawn by using the following set of the system parameters:

$$\rho = 7849\ \text{kg}, \quad A_{cs} = 7.687 \times 10^{-3}\ \text{m}^2, \quad I = 3.055 \times 10^{-5}\ \text{m}^4,$$
$$E = 2 \times 10^{11}\ \text{N/m}^2, \quad k_f = 10^8\ \text{N/m}^2 \tag{10}$$

This set describes a realistic rail and a statically measured stiffness of the subsoil.

Thus, the natural frequency of the mass must lie within the domain bounded by the solid line in Fig. 2. As follows from the characteristic Eq. (6), this frequency depends on the velocity and the magnitude of the mass. The dependence of the natural frequency on the velocity, calculated in accordance with the characteristic Eq. (6) is presented in Fig. 2 for two different magnitudes of the mass, namely for $m = 1000$ kg and $m = 2000$ kg. The figure shows that the smaller the mass, the closer is the natural frequency to the critical frequency $\Omega^*(V)$ (to the bifurcation curve). The larger the mass, the smaller is its natural frequency.

Fig. 2 clearly shows that if the velocity of the mass is smaller than the minimum phase velocity $V_{ph}^{\min}$ of waves in the beam, which is given by the crossing point of the bifurcation curve with the vertical axis (approximately 900 m/s), then the natural vibrations of the mass are harmonic. Of course, for the mass to start vibrating harmonically a certain time is needed in order for the transient oscillations related to the initial conditions to have disappeared. Thus, it is possible to declare that if $V < V_{ph}^{\min}$, then in the limit $t \to \infty$ vibrations of a mass that uniformly moves along the beam on the homogeneous elastic foundation can be described as

$$u_0^{(0)}(t) = A\exp(\mathrm{i}\Omega t) + B\exp(-\mathrm{i}\Omega t), \tag{11}$$

with Ω the natural frequency (real) and A, B as unknown constants. Deflection of the beam that corresponds to these vibrations of the mass can be found from the system of Eq. (4) by looking for the solution $w^{(0)}(x,t)$ in the following form:

$$w^{(0)}(x,t) = \sum_n \left(C_{An}\exp(\mathrm{i}\Omega t)\exp(\mathrm{i}k_n^A(Vt-x)) + C_{Bn}\exp(-\mathrm{i}\Omega t)\exp(\mathrm{i}k_n^B(Vt-x))\right). \tag{12}$$

In expression (12), the subscripts and superscripts A and B show that the vibrations of the beam correspond to the mass vibrations of the form $A\exp(\mathrm{i}\Omega t)$ and $B\exp(-\mathrm{i}\Omega t)$, respectively. Further, C_{An} and C_{Bn} are unknown constants, while k_n^A and k_n^B are complex wavenumbers that satisfy the dispersion Eq. (8).

Substituting expression (12) into the system of Eq. (4), the unknown constants C_{An} and C_{Bn} can be found to give

$$w^{(0)}(x,t) = \begin{cases} \mathrm{e}^{\mathrm{i}\Omega t}(C_{A1}^{+}\mathrm{e}^{\mathrm{i}k_1^A(Vt-x)} + C_{A2}^{+}\mathrm{e}^{\mathrm{i}k_2^A(Vt-x)}) + \mathrm{e}^{-\mathrm{i}\Omega t}(C_{B1}^{+}\mathrm{e}^{\mathrm{i}k_1^B(Vt-x)} + C_{B2}^{+}\mathrm{e}^{\mathrm{i}k_2^B(Vt-x)}), & x \geqslant Vt, \\ \mathrm{e}^{\mathrm{i}\Omega t}(C_{A1}^{-}\mathrm{e}^{\mathrm{i}k_3^A(Vt-x)} + C_{A2}^{-}\mathrm{e}^{\mathrm{i}k_4^A(Vt-x)}) + \mathrm{e}^{-\mathrm{i}\Omega t}(C_{B1}^{-}\mathrm{e}^{\mathrm{i}k_3^B(Vt-x)} + C_{B2}^{-}\mathrm{e}^{\mathrm{i}k_4^B(Vt-x)}), & x \leqslant Vt. \end{cases} \tag{13}$$

In this expression, $\mathrm{Im}(k_{1,2}^{A,B}) < 0$, $\mathrm{Im}(k_{3,4}^{A,B}) > 0$, which fulfils the condition that the beam deflection should vanish as $|x - Vt| \to \infty$. The constants $C_{A1,A2,B1,B2}^{\pm}$ are given in Appendix A. Expression (13) describes a deflection field in the beam that moves together with the mass and decays exponentially (having spatial oscillations) with the distance from the mass.

Thus, the unperturbed problem (4) has been solved, and the solution is given by expressions (11) and (13). The effect of the small inhomogeneity and small viscosity can be analysed by solving the system of Eq. (5). This is accomplished in the next section, whose title involves the term "non-resonance case" due to the reasons that will become clear later.

4. Perturbation analysis in the non-resonance case

In this section the system of Eq. (5) is studied that determines the influence of the small inhomogeneity of the foundation on vibrations of the mass and the beam. The viscosity of the foundation is temporarily assumed to be equal to zero, e.g., $v_f = 0$. Substituting the unperturbed solutions (11) and (13) into this system, and representing $\cos(\chi x)$ as $(\exp(\mathrm{i}\chi x) + \exp(-\mathrm{i}\chi x))/2$, the following equations of motion for the beam before and behind the mass are obtained:

For $x > Vt$,

$$\begin{aligned}
&\rho A_{cs}\frac{\partial^2 w^{(1)}}{\partial t^2} + EI\frac{\partial^4 w^{(1)}}{\partial x^4} + k_f w^{(1)} \\
&= -\frac{k_f}{2}\mathrm{e}^{\mathrm{i}\Omega t}(C_{A1}^{+}\mathrm{e}^{\mathrm{i}k_1^A Vt - \mathrm{i}x(k_1^A+\chi)} + C_{A1}^{+}\mathrm{e}^{\mathrm{i}k_1^A Vt - \mathrm{i}x(k_1^A-\chi)} + C_{A2}^{+}\mathrm{e}^{\mathrm{i}k_2^A Vt - \mathrm{i}x(k_2^A+\chi)} + C_{A2}^{+}\mathrm{e}^{\mathrm{i}k_2^A Vt - \mathrm{i}x(k_2^A-\chi)}) \\
&- \frac{k_f}{2}\mathrm{e}^{-\mathrm{i}\Omega t}(C_{B1}^{+}\mathrm{e}^{\mathrm{i}k_1^B Vt - \mathrm{i}x(k_1^B+\chi)} + C_{B1}^{+}\mathrm{e}^{\mathrm{i}k_1^B Vt - \mathrm{i}x(k_1^B-\chi)} + C_{B2}^{+}\mathrm{e}^{\mathrm{i}k_2^B Vt - \mathrm{i}x(k_2^B+\chi)} + C_{B2}^{+}\mathrm{e}^{\mathrm{i}k_2^B Vt - \mathrm{i}x(k_2^B-\chi)}).
\end{aligned} \tag{14}$$

For $x < Vt$,

$$\begin{aligned}
&\rho A_{cs}\frac{\partial^2 w^{(1)}}{\partial t^2} + EI\frac{\partial^4 w^{(1)}}{\partial x^4} + k_f w^{(1)} \\
&= -\frac{k_f}{2}\mathrm{e}^{\mathrm{i}\Omega t}(C_{A1}^{-}\mathrm{e}^{\mathrm{i}k_3^A Vt - \mathrm{i}x(k_3^A+\chi)} + C_{A1}^{-}\mathrm{e}^{\mathrm{i}k_3^A Vt - \mathrm{i}x(k_3^A-\chi)} + C_{A2}^{-}\mathrm{e}^{\mathrm{i}k_4^A Vt - \mathrm{i}x(k_4^A+\chi)} + C_{A2}^{-}\mathrm{e}^{\mathrm{i}k_4^A Vt - \mathrm{i}x(k_4^A-\chi)}) \\
&- \frac{k_f}{2}\mathrm{e}^{-\mathrm{i}\Omega t}(C_{B1}^{-}\mathrm{e}^{\mathrm{i}k_3^B Vt - \mathrm{i}x(k_3^B+\chi)} + C_{B1}^{-}\mathrm{e}^{\mathrm{i}k_3^B Vt - \mathrm{i}x(k_3^B-\chi)} + C_{B2}^{-}\mathrm{e}^{\mathrm{i}k_4^B Vt - \mathrm{i}x(k_4^B+\chi)} + C_{B2}^{-}\mathrm{e}^{\mathrm{i}k_4^B Vt - \mathrm{i}x(k_4^B-\chi)}).
\end{aligned} \tag{15}$$

The boundary conditions at $x = Vt$ remain unchanged:

$$\begin{aligned}
&[w^{(1)}]_{x=Vt} = \left[\frac{\partial w^{(1)}}{\partial x}\right]_{x=Vt} = \left[\frac{\partial^2 w^{(1)}}{\partial x^2}\right]_{x=Vt} = 0, \\
&w^{(1)}\big|_{x=Vt} = w_0^{(1)}, \\
&EI\left[\frac{\partial^3 w^{(1)}}{\partial x^3}\right]_{x=Vt} = -m\frac{\mathrm{d}^2 w_0^{(1)}}{\mathrm{d}t^2}.
\end{aligned} \tag{16}$$

To solve the system of Eqs. (14)–(16), it is customary to look for the solution in the following form:

$$w^{(1)} = w_{free}^{(1)} + w_{forced}^{(1)} \tag{17}$$

with $w_{forced}^{(1)}$ the forced solution to Eqs. (14) and (15). This forced solution describes the effect of the inhomogeneity on the deflection field in the beam that is generated by the moving and vibrating mass. The expression for $w_{forced}^{(1)}$ can be found straightforwardly to give

For $x > Vt$:

$$w^{(1)}_{forced} = e^{i\Omega t}(C^{+}_{11}e^{ik^A_1 Vt - ix(k^A_1+\chi)} + C^{+}_{12}e^{ik^A_1 Vt - ix(k^A_1-\chi)} + C^{+}_{21}e^{ik^A_2 Vt - ix(k^A_2+\chi)} + C^{+}_{22}e^{ik^A_2 Vt - ix(k^A_2-\chi)})$$
$$+ e^{-i\Omega t}(C^{+}_{31}e^{ik^B_1 Vt - ix(k^B_1+\chi)} + C^{+}_{32}e^{ik^B_1 Vt - ix(k^B_1-\chi)} + C^{+}_{41}e^{ik^B_2 Vt - ix(k^B_2+\chi)} + C^{+}_{42}e^{ik^B_2 Vt - ix(k^B_2-\chi)}). \quad (18)$$

For $x < Vt$,

$$w^{(1)}_{forced} = e^{i\Omega t}(C^{-}_{11}e^{ik^A_3 Vt - ix(k^A_3+\chi)} + C^{-}_{12}e^{ik^A_3 Vt - ix(k^A_3-\chi)} + C^{-}_{21}e^{ik^A_4 Vt - ix(k^A_4+\chi)} + C^{-}_{22}e^{ik^A_4 Vt - ix(k^A_4-\chi)})$$
$$+ e^{-i\Omega t}(C^{-}_{31}e^{ik^B_3 Vt - ix(k^B_3+\chi)} + C^{-}_{32}e^{ik^B_3 Vt - ix(k^B_3-\chi)} + C^{-}_{41}e^{ik^B_4 Vt - ix(k^B_4+\chi)} + C^{-}_{42}e^{ik^B_4 Vt - ix(k^B_4-\chi)}) \quad (19)$$

with the constants $C^{\pm}_{ij}$, $i = 1, \ldots, 4$, $j = 1, 2$ defined in Appendix A.

Substituting (17) into the system of Eqs. (14)–(16), taking into account that $w^{(1)}_{forced}$ is the solution to Eqs. (14) and (15), and making use of solutions (18) and (19), the following system of equations is obtained with respect to $w^{(1)}_{free}$:

$$\rho A_{cs}\frac{\partial^2 w^{(1)}_{free}}{\partial t^2} + EI\frac{\partial^4 w^{(1)}_{free}}{\partial x^4} + k_f w^{(1)}_{free} = 0,$$
$$[w^{(1)}_{free}]_{x=Vt} = 0,$$
$$\left[\frac{\partial}{\partial x}w^{(1)}_{free}\right]_{x=Vt} = D_{11}e^{it(\Omega-\chi V)} + D_{12}e^{it(\Omega+\chi V)} + D_{13}e^{-it(\Omega+\chi V)} + D_{14}e^{-it(\Omega-\chi V)},$$
$$\left[\frac{\partial^2}{\partial x^2}w^{(1)}_{free}\right]_{x=Vt} = D_{21}e^{it(\Omega-\chi V)} + D_{22}e^{it(\Omega+\chi V)} + D_{23}e^{-it(\Omega+\chi V)} + D_{24}e^{-it(\Omega-\chi V)},$$
$$w^{(1)}_{free}\Big|_{x=Vt} = w^{(1)}_0 + D_{31}e^{it(\Omega-\chi V)} + D_{32}e^{it(\Omega+\chi V)} + D_{33}e^{-it(\Omega+\chi V)} + D_{34}e^{-it(\Omega-\chi V)},$$
$$\left[\frac{\partial^3}{\partial x^3}w^{(1)}_{free}\right]_{x=Vt} = -\frac{m}{EI}\ddot{w}^{(1)}_0 + D_{41}e^{it(\Omega-\chi V)} + D_{42}e^{it(\Omega+\chi V)} + D_{43}e^{-it(\Omega+\chi V)} + D_{44}e^{-it(\Omega-\chi V)}. \quad (20)$$

It is easy to see that the system of Eq. (20) is analogous to the system of Eq. (4), which describes the vibrations of the mass on the homogeneous beam. The only difference between these two systems is that the boundary conditions at $x = Vt$ contain the right sides that describe "forces" that act in the contact force because of the inhomogeneity of the foundation. All these "forces" are harmonic and have frequencies equal to $\pm(\Omega \pm V\chi)$. Thus, since it is known that the natural frequency of the mass on the homogeneous beam is equal to Ω, it can be concluded that resonance ($w^{(1)}_{free} \to \infty$) will take place in the system if one of the following four equations is satisfied:

$$\Omega = \pm(\Omega \pm V\chi) \quad (21)$$

Only one of the Eqs. (21) can be satisfied, namely the equation $\Omega = -\Omega + V\chi$ (the other equations cannot be satisfied since Ω, V and χ are positive values). The solution to this equation is

$$V\chi = 2\Omega. \quad (22)$$

Thus, if relation (22) were to be satisfied, then the solution $w^{(1)}_{free}$ would tend to infinity. This would imply that $w^{(0)}(x,t) \ll \mu w^{(1)}(x,t)$ and, therefore, the original assumption that the small inhomogeneity provides a small variation of the unperturbed solution is violated. As a consequence, series (3) become divergent.

The conclusion, which has to be drawn from this fact, is that the perturbation technique based on representation (3) is not applicable once relation (22) is satisfied. In Ref. [21], however, in the analysis of the Mathieu equation, it is shown that the perturbation technique can be modified to give a relevant solution for the case when relation (22) is satisfied. In this book, this case was referred to as resonance case, which is a reasonable terminology to be used.

The modification of the perturbation method is presented in the next section. Before starting with this section, however, it is worth noting that relation (22) is fully analogous to the condition of the parametric resonance in the Mathieu's equation [21,22]. This equation describes, for example, vibrations of a mass on a spring, whose stiffness varies harmonically in time. If the Mathieu's equation is written in the form

$$\ddot{x} + \omega_0^2 x(1 + \mu\cos(\omega_p t)) = 0, \tag{23}$$

then the condition for the parametric resonance (more precisely, for the first zone of the resonance) is given as

$$2\omega_0 \approx \omega_p. \tag{24}$$

Condition (24) implies that the parametric resonance (that contains in an exponential increase of the amplitude of vibrations) occurs if the doubled natural frequency of the unperturbed mass–spring system is approximately equal to the frequency of the variation of the spring stiffness.

The analogy between conditions (22) and (24) is evident. Indeed, the natural frequency Ω of the mass vibrations on the homogeneous beam is the direct analogy to ω_0, whereas the frequency χV represents the frequency of variation of the stiffness of the elastic foundation under the moving mass.

Taking this analogy into account it is natural to expect that vibrations of the moving mass on the periodically inhomogeneous beam can become unstable due to the parametric resonance as happens in the systems described by the Mathieu's equation. Correctness of this expectation is proven in the following section.

5. Perturbation analysis in the resonance case

In this section, the original problem (2) is studied in the resonance case, in which the relation (22) is approximately satisfied, e.g., the following relation holds

$$\chi = \frac{2(\Omega + \mu\delta)}{V}, \tag{25}$$

with $\mu\delta \ll \Omega$ a small mistuning.

The study is accomplished in the following manner. First, as in the previous section, the results are obtained for the undamped case $v_f = 0$. The viscosity is taken into account later giving a generalisation of the results.

By analogy with the principle used in [21] for the analysis of the parametric resonance in the Mathieu's equation, a solution to problem (2) is sought for in the following form:

$$u_0(t) = A(\mu t)\mathrm{e}^{\mathrm{i}t(\Omega+\mu\delta)} + B(\mu t)\mathrm{e}^{-\mathrm{i}t(\Omega+\mu\delta)} + \mu w_0^{(1)}(t),$$

$$w(x,t) = \mu w^{(1)}(x,t) + \begin{cases} \mathrm{e}^{\mathrm{i}t(\Omega+\mu\delta)}(C_{A1}^{+}(\mu x,\mu t)\mathrm{e}^{\mathrm{i}k_1^A(Vt-x)} + C_{A2}^{+}(\mu x,\mu t)\mathrm{e}^{\mathrm{i}k_2^A(Vt-x)}) & \\ +\mathrm{e}^{-\mathrm{i}t(\Omega+\mu\delta)}(C_{B1}^{+}(\mu x,\mu t)\mathrm{e}^{\mathrm{i}k_1^B(Vt-x)} + C_{B2}^{+}(\mu x,\mu t)\mathrm{e}^{\mathrm{i}k_2^B(Vt-x)}), & x > Vt, \\ \mathrm{e}^{\mathrm{i}t(\Omega+\mu\delta)}(C_{A1}^{-}(\mu x,\mu t)\mathrm{e}^{\mathrm{i}k_3^A(Vt-x)} + C_{A2}^{-}(\mu x,\mu t)\mathrm{e}^{\mathrm{i}k_4^A(Vt-x)}) & \\ +\mathrm{e}^{-\mathrm{i}t(\Omega+\mu\delta)}(C_{B1}^{-}(\mu x,\mu t)\mathrm{e}^{\mathrm{i}k_3^B(Vt-x)} + C_{B2}^{-}(\mu x,\mu t)e^{\mathrm{i}k_4^B(Vt-x)}), & x < Vt, \end{cases} \tag{26}$$

with $k_{1,2,3,4}^{A,B}$ the roots of the dispersion Eq. (8).

Expressions (26) are similar to solution (3) (in which expressions (11) and (13) are substituted) with the only difference being that the amplitudes of waves and vibrations are assumed to have a weak dependence on time and the spatial co-ordinate. It will be seen that introduction of this dependence constrains the perturbation terms $\mu w_0^{(1)}(t)$ and $\mu w^{(1)}(x,t)$ to be much smaller than the modified unperturbed solution. Actually, the following analysis is based on the principle of finding the slowly varying amplitudes for the unperturbed solution so that the perturbation terms remain small. Fulfilling this requirement, a relation between the system parameters can be found that corresponds to a *slow* increase of the amplitude of the system vibrations in time, e.g. to the parametric resonance.

Substituting expressions (26) into the system of Eq. (2) and collecting the terms of the order μ^0, the system of equations is obtained that is presented in Appendix B. As shown in this appendix, this system of equations is satisfied independently of the choice of the amplitudes $C_{Aj}^{\pm}$ and $C_{Bj}^{\pm}$.

Collecting the terms of the order μ^1, the following system of equations is obtained:

For $x > Vt$,

$$\begin{aligned} &\rho A_{cs}\frac{\partial^2 w^{(1)}}{\partial t^2} + EI\frac{\partial^4 w^{(1)}}{\partial x^4} + k_f w^{(1)} \\ &\quad= -k_f\cos(\chi x)\mathrm{e}^{\mathrm{i}t(\Omega+\mu\delta)}(C_{A1}^{+}(\mu x,\mu t)\mathrm{e}^{\mathrm{i}k_1^A(Vt-x)} + C_{A2}^{+}(\mu x,\mu t)\mathrm{e}^{\mathrm{i}k_2^A(Vt-x)}) \\ &\quad - k_f\cos(\chi x)\mathrm{e}^{-\mathrm{i}t(\Omega+\mu\delta)}\left(C_{B1}^{+}(\mu x,\mu t)\mathrm{e}^{\mathrm{i}k_1^B(Vt-x)} + C_{B2}^{+}(\mu x,\mu t)\mathrm{e}^{\mathrm{i}k_2^B(Vt-x)}\right) \\ &\quad - \left(2\rho A_{cs}(Vk_1^A+\Omega)\left(\mathrm{i}\frac{\partial C_{A1}^{+}}{\partial(\mu t)} - \delta C_{A1}^{+}\right) + 4\mathrm{i}EI(k_1^A)^3\frac{\partial C_{A1}^{+}}{\partial(\mu x)}\right)\exp(\mathrm{i}k_1^A(Vt-x) + \mathrm{i}t(\Omega+\mu\delta)) \\ &\quad - \left(2\rho A_{cs}(Vk_2^A+\Omega)\left(\mathrm{i}\frac{\partial C_{A2}^{+}}{\partial(\mu t)} - \delta C_{A2}^{+}\right) + 4\mathrm{i}EI(k_2^A)^3\frac{\partial C_{A2}^{+}}{\partial(\mu x)}\right)\exp(\mathrm{i}k_2^A(Vt-x) + \mathrm{i}t(\Omega+\mu\delta)) \\ &\quad - \left(2\rho A_{cs}(Vk_1^B-\Omega)\left(\mathrm{i}\frac{\partial C_{B1}^{+}}{\partial(\mu t)} + \delta C_{B1}^{+}\right) + 4\mathrm{i}EI(k_1^B)^3\frac{\partial C_{B1}^{+}}{\partial(\mu x)}\right)\exp(\mathrm{i}k_1^B(Vt-x) - \mathrm{i}t(\Omega+\mu\delta)) \\ &\quad - \left(2\rho A_{cs}(Vk_2^B-\Omega)\left(\mathrm{i}\frac{\partial C_{B2}^{+}}{\partial(\mu t)} + \delta C_{B2}^{+}\right) + 4\mathrm{i}EI(k_2^B)^3\frac{\partial C_{B2}^{+}}{\partial(\mu x)}\right)\exp(\mathrm{i}k_2^B(Vt-x) - \mathrm{i}t(\Omega+\mu\delta)). \end{aligned} \tag{27}$$

For $x < Vt$,

$$
\begin{aligned}
&\rho A_{cs}\frac{\partial^2 w^{(1)}}{\partial t^2} + EI\frac{\partial^4 w^{(1)}}{\partial x^4} + k_f w^{(1)} \\
&= -k_f\cos(\chi x)\mathrm{e}^{\mathrm{i}t(\Omega+\mu\delta)}(C_{A1}^-(\mu x,\mu t)\mathrm{e}^{\mathrm{i}k_3^A(Vt-x)} + C_{A2}^-(\mu x,\mu t)\mathrm{e}^{\mathrm{i}k_4^A(Vt-x)}) \\
&- k_f\cos(\chi x)\mathrm{e}^{-\mathrm{i}t(\Omega+\mu\delta)}(C_{B1}^-(\mu x,\mu t)\mathrm{e}^{\mathrm{i}k_3^B(Vt-x)} + C_{B2}^-(\mu x,\mu t)\mathrm{e}^{\mathrm{i}k_4^B(Vt-x)}) \\
&- \left(2\rho A_{cs}(Vk_3^A+\Omega)\left(\mathrm{i}\frac{\partial C_{A1}^-}{\partial(\mu t)} - \delta C_{A1}^-\right) + 4\mathrm{i}EI(k_3^A)^3\frac{\partial C_{A1}^-}{\partial(\mu x)}\right)\exp(\mathrm{i}k_3^A(Vt-x) + \mathrm{i}t(\Omega+\mu\delta)) \\
&- \left(2\rho A_{cs}(Vk_4^A+\Omega)\left(\mathrm{i}\frac{\partial C_{A2}^-}{\partial(\mu t)} - \delta C_{A2}^-\right) + 4\mathrm{i}EI(k_4^A)^3\frac{\partial C_{A2}^-}{\partial(\mu x)}\right)\exp(\mathrm{i}k_4^A(Vt-x) + \mathrm{i}t(\Omega+\mu\delta)) \\
&- \left(2\rho A_{cs}(Vk_3^B-\Omega)\left(\mathrm{i}\frac{\partial C_{B1}^-}{\partial(\mu t)} + \delta C_{B1}^-\right) + 4\mathrm{i}EI(k_3^B)^3\frac{\partial C_{B1}^-}{\partial(\mu x)}\right)\exp(\mathrm{i}k_3^B(Vt-x) - \mathrm{i}t(\Omega+\mu\delta)) \\
&- \left(2\rho A_{cs}(Vk_4^B-\Omega)\left(\mathrm{i}\frac{\partial C_{B2}^-}{\partial(\mu t)} + \delta C_{B2}^-\right) + 4\mathrm{i}EI(k_4^B)^3\frac{\partial C_{B2}^-}{\partial(\mu x)}\right)\exp(\mathrm{i}k_4^B(Vt-x) - \mathrm{i}t(\Omega+\mu\delta)) \quad (28)
\end{aligned}
$$

For $x = Vt$,

$$[w^{(1)}]_{x=Vt} = 0, \quad (29)$$

$$
\begin{aligned}
\left[\frac{\partial w^{(1)}}{\partial x}\right]_{x=Vt} = &- \mathrm{e}^{\mathrm{i}t(\Omega+\mu\delta)}\left(\frac{\partial C_{A1}^+}{\partial(\mu x)} + \frac{\partial C_{A2}^+}{\partial(\mu x)} - \frac{\partial C_{A1}^-}{\partial(\mu x)} - \frac{\partial C_{A2}^-}{\partial(\mu x)}\right)_{x=Vt} \\
&- \mathrm{e}^{-\mathrm{i}t(\Omega+\mu\delta)}\left(\frac{\partial C_{B1}^+}{\partial(\mu x)} + \frac{\partial C_{B2}^+}{\partial(\mu x)} - \frac{\partial C_{B1}^-}{\partial(\mu x)} - \frac{\partial C_{B2}^-}{\partial(\mu x)}\right)_{x=Vt}, \quad (30)
\end{aligned}
$$

$$
\begin{aligned}
\left[\frac{\partial^2 w^{(1)}}{\partial x^2}\right]_{x=Vt} = &2\mathrm{i}\mathrm{e}^{\mathrm{i}t(\Omega+\mu\delta)}\left(k_1^A\frac{\partial C_{A1}^+}{\partial(\mu x)} + k_2^A\frac{\partial C_{A2}^+}{\partial(\mu x)} - k_3^A\frac{\partial C_{A1}^-}{\partial(\mu x)} - k_4^A\frac{\partial C_{A2}^-}{\partial(\mu x)}\right)_{x=Vt} \\
&+ 2\mathrm{i}\mathrm{e}^{-\mathrm{i}t(\Omega+\mu\delta)}\left(k_1^B\frac{\partial C_{B1}^+}{\partial(\mu x)} + k_2^B\frac{\partial C_{B2}^+}{\partial(\mu x)} - k_3^B\frac{\partial C_{B1}^-}{\partial(\mu x)} - k_4^B\frac{\partial C_{B2}^-}{\partial(\mu x)}\right)_{x=Vt}, \quad (31)
\end{aligned}
$$

$$w^{(1)}(Vt,t) = w_0^{(1)}(t), \quad (32)$$

$$
\begin{aligned}
EI\left[\frac{\partial^3 w^{(1)}}{\partial x^3}\right]_{x=Vt} = &- m\frac{\mathrm{d}^2 w_0^{(1)}}{\mathrm{d}t^2} + \mathrm{e}^{\mathrm{i}t(\Omega+\mu\delta)}\Bigg(-2m\Omega\left(\mathrm{i}\frac{\partial A}{\partial(\mu t)} - \delta A\right) \\
&+ 3EI\left((k_1^A)^2\frac{\partial C_{A1}^+}{\partial(\mu x)} + (k_2^A)^2\frac{\partial C_{A2}^+}{\partial(\mu x)} - (k_3^A)^2\frac{\partial C_{A1}^-}{\partial(\mu x)} - (k_4^A)^2\frac{\partial C_{A2}^-}{\partial(\mu x)}\right)\Bigg)_{x=Vt} \\
&+ \mathrm{e}^{-\mathrm{i}t(\Omega+\mu\delta)}\Bigg(2m\Omega\left(\mathrm{i}\frac{\partial B}{\partial(\mu t)} + \delta B\right) \\
&+ 3EI\left((k_1^B)^2\frac{\partial C_{B1}^+}{\partial(\mu x)} + (k_2^B)^2\frac{\partial C_{B2}^+}{\partial(\mu x)} - (k_3^B)^2\frac{\partial C_{B1}^-}{\partial(\mu x)} - (k_4^B)^2\frac{\partial C_{B2}^-}{\partial(\mu x)}\right)\Bigg)_{x=Vt}. \quad (33)
\end{aligned}
$$

For the perturbation method to be applicable, the perturbation terms $w^{(1)}(x,t)$ and $w_0^{(1)}(t)$ should be prohibited from increasing with time. To achieve this, all forces that act on the beam and on the mass and which can cause resonance must be set to zero. There are two types of forces that disturb the system: the distributed ones that stay on the right-hand side of Eqs. (27) and (28), and the concentrated ones that enter the boundary condition (33). Note that the external moment in the boundary condition (31) cannot activate the heave vibrations of the mass.

Consider first the distributed forces in Eqs. (27) and (28). It is obvious that the last four terms on the right-hand side of these equations would cause the resonance response since they are proportional to the normal waves in the beam: $\exp(\pm \mathrm{i}\Omega t)\exp(\mathrm{i}k^{A,B}_{1,2,3,4}(Vt-x))$. Thus, it is necessary that these terms vanish, e.g.,

$$
\begin{aligned}
&2\rho A_{cs}(Vk_1^A+\Omega)\left(\mathrm{i}\frac{\partial C^+_{A1}}{\partial(\mu t)}-\delta C^+_{A1}\right)+4\mathrm{i}EI(k_1^A)^3\frac{\partial C^+_{A1}}{\partial(\mu x)}=0,\\
&2\rho A_{cs}(Vk_2^A+\Omega)\left(\mathrm{i}\frac{\partial C^+_{A2}}{\partial(\mu t)}-\delta C^+_{A2}\right)+4\mathrm{i}EI(k_2^A)^3\frac{\partial C^+_{A2}}{\partial(\mu x)}=0,\\
&2\rho A_{cs}(Vk_1^B+\Omega)\left(\mathrm{i}\frac{\partial C^+_{B1}}{\partial(\mu t)}+\delta C^+_{B1}\right)+4\mathrm{i}EI(k_1^B)^3\frac{\partial C^+_{B1}}{\partial(\mu x)}=0,\\
&2\rho A_{cs}(Vk_2^B-\Omega)\left(\mathrm{i}\frac{\partial C^+_{B2}}{\partial(\mu t)}+\delta C^+_{B2}\right)+4\mathrm{i}EI(k_2^B)^3\frac{\partial C^+_{B2}}{\partial(\mu x)}=0,\\
&2\rho A_{cs}(Vk_3^A+\Omega)\left(\mathrm{i}\frac{\partial C^-_{A1}}{\partial(\mu t)}-\delta C^-_{A1}\right)+4\mathrm{i}EI(k_3^A)^3\frac{\partial C^-_{A1}}{\partial(\mu x)}=0,\\
&2\rho A_{cs}(Vk_4^A+\Omega)\left(\mathrm{i}\frac{\partial C^-_{A2}}{\partial(\mu t)}-\delta C^-_{A2}\right)+4\mathrm{i}EI(k_4^A)^3\frac{\partial C^-_{A2}}{\partial(\mu x)}=0,\\
&2\rho A_{cs}(Vk_3^B-\Omega)\left(\mathrm{i}\frac{\partial C^-_{B1}}{\partial(\mu t)}+\delta C^-_{B1}\right)+4\mathrm{i}EI(k_3^B)^3\frac{\partial C^-_{B1}}{\partial(\mu x)}=0,\\
&2\rho A_{cs}(Vk_4^B-\Omega)\left(\mathrm{i}\frac{\partial C^-_{B2}}{\partial(\mu t)}+\delta C^-_{B2}\right)+4\mathrm{i}EI(k_4^B)^3\frac{\partial C^-_{B2}}{\partial(\mu x)}=0.
\end{aligned}
\tag{34}
$$

Thus, the distributed forces that could cause the increase of the beam vibrations have been required to vanish. However, some of the remaining forces on the right-hand of Eqs. (27) and (28) could also lead to resonance in the system. These are the forces whose frequency equals to $(\Omega+\mu\delta)$ in the contact point. The other forces cannot lead to resonance and, therefore, are not relevant for the further analysis that is aimed at finding the conditions under which the perturbation terms do not increase.

Considering relations (34) satisfied, and neglecting the non-resonance terms on the right sides (the terms whose frequency is not equal to $(\Omega+\mu\delta)$ at $x=Vt$), Eqs. (27) and (28) are rewritten as follows

For $x>Vt$,

$$
\begin{aligned}
&\rho A_{cs}\frac{\partial^2 w^{(1)}}{\partial t^2}+EI\frac{\partial^4 w^{(1)}}{\partial x^4}+k_f w^{(1)}\\
&\quad=-\frac{k_f}{2}\exp(-\mathrm{i}\chi x)\mathrm{e}^{\mathrm{i}t(\Omega+\mu\delta)}(C^+_{A1}(\mu x,\mu t)\mathrm{e}^{\mathrm{i}k_1^A(Vt-x)}+C^+_{A2}(\mu x,\mu t)\mathrm{e}^{\mathrm{i}k_2^A(Vt-x)})\\
&\qquad-\frac{k_f}{2}\exp(\mathrm{i}\chi x)\mathrm{e}^{-\mathrm{i}t(\Omega+\mu\delta)}(C^+_{B1}(\mu x,\mu t)\mathrm{e}^{\mathrm{i}k_1^B(Vt-x)}+C^+_{B2}(\mu x,\mu t)\mathrm{e}^{\mathrm{i}k_2^B(Vt-x)}).
\end{aligned}
\tag{35}
$$

For $x < Vt$,

$$\rho A_{cs}\frac{\partial^2 w^{(1)}}{\partial t^2} + EI\frac{\partial^4 w^{(1)}}{\partial x^4} + k_f w^{(1)}$$
$$= -\frac{k_f}{2}\exp(-i\chi x)\mathrm{e}^{\mathrm{i}t(\Omega+\mu\delta)}(C^-_{A1}(\mu x,\mu t)\mathrm{e}^{\mathrm{i}k_3^A(Vt-x)} + C^-_{A2}(\mu x,\mu t)\mathrm{e}^{\mathrm{i}k_4^A(Vt-x)})$$
$$-\frac{k_f}{2}\exp(\mathrm{i}\chi x)\mathrm{e}^{-\mathrm{i}t(\Omega+\mu\delta)}(C^-_{B1}(\mu x,\mu t)\mathrm{e}^{\mathrm{i}k_3^B(Vt-x)} + C^-_{B2}(\mu x,\mu t)\mathrm{e}^{\mathrm{i}k_4^B(Vt-x)}). \quad (36)$$

It is customary to seek for the solution of these equations in the form

$$w^{(1)} = w^{(1)}_{free} + w^{(1)}_{forced} \quad (37)$$

with $w^{(1)}_{forced}$ the forced solution of Eqs. (27) and (28) that describes the effect of the foundation inhomogeneity on the deflection field, which the mass perturbs in the beam. This solution reads:

For $x > Vt$,

$$w^{(1)}_{forced} = \mathrm{e}^{\mathrm{i}t(\Omega+\mu\delta)-\mathrm{i}\chi x}(\tilde{C}^+_{11}(\mu x,\mu t)\mathrm{e}^{\mathrm{i}k_1^A(Vt-x)} + \tilde{C}^+_{12}(\mu x,\mu t)\mathrm{e}^{\mathrm{i}k_2^A(Vt-x)})$$
$$+\mathrm{e}^{-\mathrm{i}t(\Omega+\mu\delta)+\mathrm{i}\chi x}(\tilde{C}^+_{21}(\mu x,\mu t)\mathrm{e}^{\mathrm{i}k_1^B(Vt-x)} + \tilde{C}^+_{22}(\mu x,\mu t)\mathrm{e}^{\mathrm{i}k_2^B(Vt-x)}). \quad (38)$$

For $x < Vt$,

$$w^{(1)}_{forced} = \mathrm{e}^{\mathrm{i}t(\Omega+\mu\delta)-\mathrm{i}\chi x}(\tilde{C}^-_{11}(\mu x,\mu t)\mathrm{e}^{\mathrm{i}k_3^A(Vt-x)} + \tilde{C}^-_{12}(\mu x,\mu t)\mathrm{e}^{\mathrm{i}k_4^A(Vt-x)})$$
$$+\mathrm{e}^{-\mathrm{i}t(\Omega+\mu\delta)+i\chi x}(\tilde{C}^-_{21}(\mu x,\mu t)\mathrm{e}^{\mathrm{i}k_3^B(Vt-x)} + \tilde{C}^-_{22}(\mu x,\mu t)\mathrm{e}^{\mathrm{i}k_4^B(Vt-x)}) \quad (39)$$

with the constants $\tilde{C}^{\pm}_{ij}$, $i = 1, \ldots, 4,\ j = 1, 2$ defined in Appendix C.

Substituting representation (37) and expressions (38) and (39) into the boundary condition (33), the following equation is obtained

$$EI\left[\frac{\partial^3 w^{(1)}_{free}}{\partial x^3}\right]_{x=Vt} + m\frac{\mathrm{d}^2 w_0^{(1)}}{\mathrm{d}t^2} = \mathrm{e}^{\mathrm{i}t(\Omega+\mu\delta)}\Bigg\{-2m\Omega\left(\mathrm{i}\frac{\partial A}{\partial(\mu t)} - \delta A\right)$$
$$+3EI\left((k_1^A)^2\frac{\partial C^+_{A1}}{\partial(\mu x)} + (k_2^A)^2\frac{\partial C^+_{A2}}{\partial(\mu x)} - (k_3^A)^2\frac{\partial C^-_{A1}}{\partial(\mu x)} - (k_4^A)^2\frac{\partial C^-_{A2}}{\partial(\mu x)}\right)$$
$$-EI(\mathrm{i}(k_1^B-\chi)^3\tilde{C}^+_{21} + \mathrm{i}(k_2^B-\chi)^3\tilde{C}^+_{22} - \mathrm{i}(k_3^B-\chi)^3\tilde{C}^-_{21} - \mathrm{i}(k_4^B-\chi)^3\tilde{C}^-_{22})\Bigg\}_{x=Vt}$$
$$+\mathrm{e}^{-\mathrm{i}t(\Omega+\mu\delta)}\Bigg\{2m\Omega\left(\mathrm{i}\frac{\partial B}{\partial(\mu t)} + \delta B\right)$$
$$+3EI\left((k_1^B)^2\frac{\partial C^+_{B1}}{\partial(\mu x)} + (k_2^B)^2\frac{\partial C^+_{B2}}{\partial(\mu x)} - (k_3^B)^2\frac{\partial C^-_{B1}}{\partial(\mu x)} - (k_4^B)^2\frac{\partial C^-_{B2}}{\partial(\mu x)}\right)$$
$$-EI(\mathrm{i}(k_1^A+\chi)^3\tilde{C}^+_{11} + \mathrm{i}(k_2^A+\chi)^3\tilde{C}^+_{12} - \mathrm{i}(k_3^A+\chi)^3\tilde{C}^-_{11} - \mathrm{i}(k_4^A+\chi)^3\tilde{C}^-_{12})\Bigg\}_{x=Vt}. \quad (40)$$

Both terms, which stay in the figure brackets on the right-hand of Eq. (40), should cause resonance in the system, since their frequency is equal to the natural frequency of the mass. Thus, these terms must be required to vanish, which yields the following two equations:

$$\begin{aligned}&\left(-2m\Omega\left(\mathrm{i}\frac{\partial A}{\partial(\mu t)}-\delta A\right)+3EI\left((k_1^A)^2\frac{\partial C_{A1}^+}{\partial(\mu x)}+(k_2^A)^2\frac{\partial C_{A2}^+}{\partial(\mu x)}-(k_3^A)^2\frac{\partial C_{A1}^-}{\partial(\mu x)}-(k_4^A)^2\frac{\partial C_{A2}^-}{\partial(\mu x)}\right)\right.\\&\left.-EI(i(k_1^B-\chi)^3\tilde{C}_{21}^++i(k_2^B-\chi)^3\tilde{C}_{22}^+-\mathrm{i}(k_3^B-\chi)^3\tilde{C}_{21}^--\mathrm{i}(k_4^B-\chi)^3\tilde{C}_{22}^-)\right)_{x=Vt}=0,\\&\left(2m\Omega\left(\mathrm{i}\frac{\partial B}{\partial(\mu t)}+\delta B\right)+3EI\left((k_1^B)^2\frac{\partial C_{B1}^+}{\partial(\mu x)}+(k_2^B)^2\frac{\partial C_{B2}^+}{\partial(\mu x)}-(k_3^B)^2\frac{\partial C_{B1}^-}{\partial(\mu x)}-(k_4^B)^2\frac{\partial C_{B2}^-}{\partial(\mu x)}\right)\right.\\&\left.-EI(\mathrm{i}(k_1^A+\chi)^3\tilde{C}_{11}^++\mathrm{i}(k_2^A+\chi)^3\tilde{C}_{12}^+-\mathrm{i}(k_3^A+\chi)^3\tilde{C}_{11}^--\mathrm{i}(k_4^A+\chi)^3\tilde{C}_{12}^-)\right)_{x=Vt}=0.\end{aligned}\tag{41}$$

Eqs. (34) and (41) are sufficient conditions for the perturbed solutions terms $w^{(1)}(x,t)$ and $w_0^{(1)}(t)$ not to grow in time.

The solution to these equations can be sought in the form

$$\begin{aligned}&C_{A1}^+(\mu x,\mu t)=C_{A10}^+\exp(\mu(q_1^At-p_1^Ax)), &&C_{A2}^+(\mu x,\mu t)=C_{A20}^+\exp(\mu(q_2^At-p_2^Ax)),\\&C_{B1}^+(\mu x,\mu t)=C_{B10}^+\exp(\mu(q_1^Bt-p_1^Bx)), &&C_{B2}^+(\mu x,\mu t)=C_{B20}^+\exp(\mu(q_2^Bt-p_2^Bx)),\\&C_{A1}^-(\mu x,\mu t)=C_{A10}^-\exp(\mu(q_3^At-p_3^Ax)), &&C_{A2}^-(\mu x,\mu t)=C_{A20}^-\exp(\mu(q_4^At-p_4^Ax)),\\&C_{B1}^-(\mu x,\mu t)=C_{B10}^-\exp(\mu(q_3^Bt-p_3^Bx)), &&C_{B2}^-(\mu x,\mu t)=C_{B20}^-\exp(\mu(q_4^Bt-p_4^Bx)),\\&A(\mu t)=A_0\exp(\mu st),\quad B(\mu t)=B_0\exp(\mu st).\end{aligned}\tag{42}$$

The eigenvalue s in these expressions determines the stability of the system. Should one of the eigenvalues have a positive real part, the system would become unstable. To obtain the characteristic equation with respect to s, it is customary to use Eqs. (B.3)–(B.6). Substituting expressions (42) into these equations, a set of relations (D.1) is obtained that is presented in Appendix D. Taking these relations into account and substituting expressions (42) into the Eqs. (34) and (41), the following system of two algebraic equations with respect to A_0 and B_0 can be obtained:

$$\begin{aligned}&(\mathrm{i}s-\delta)Q_1A_0+Q_4B_0=0,\\&-Q_3A_0+Q_2(\mathrm{i}s+\delta)B_0=0.\end{aligned}\tag{43}$$

The characteristic equation is obtained from the system of Eqs. (43) by setting the determinant of this system to zero. This yields

$$s^2=-\delta^2+\frac{Q_3Q_4}{Q_1Q_2}.\tag{44}$$

It can be shown that the ratio $(Q_3Q_4)/(Q_1Q_2)$ is real and positive in the case under consideration ($V<V_{ph}^{\min}$). Therefore, the criterion for the instability (parametric resonance) to occur is that s^2 is real. Thus, the vibrations of the system are unstable if the following inequality is satisfied:

$$-\delta^2+\frac{Q_3Q_4}{Q_1Q_2}>0,$$

This inequality can be rewritten by using the resonance condition (25) as

$$|\chi V - 2\Omega| < 2\mu\sqrt{\frac{Q_3 Q_4}{Q_1 Q_2}}. \tag{45}$$

If the viscosity of the foundation v_f is not equal to zero, exactly the same procedure can be employed to obtain the characteristic equation. This equation then takes the form

$$s^2 - \mathrm{i}\left(\frac{Q_5}{Q_1} + \frac{Q_6}{Q_2}\right)s - \delta\left(\frac{Q_5}{Q_1} - \frac{Q_6}{Q_2}\right) + \delta^2 - \frac{Q_3 Q_4}{Q_1 Q_2} - \frac{Q_5 Q_6}{Q_1 Q_2} = 0, \tag{46}$$

with the same expressions for $Q_{1,2,3,4}$ that are used in Eq. (44) and the constants $Q_{5,6}$ defined in Appendix D.

The criterion for the instability in this case is that one of the roots of the characteristic equation has a positive real part. It can be shown that this criterion leads to the following system of inequalities, which being satisfied leads to vibrational instability:

$$- Q_2^2 Q_5^2 - Q_1^2 Q_6^2 + 2Q_1 Q_2 Q_5 Q_6 + 4Q_1 Q_2 Q_3 Q_4 - 4\delta(\delta Q_1 Q_2 - Q_5 Q_2 + Q_1 Q_6) > 0,$$

$$\frac{Q_6}{2Q_2} + \frac{Q_5}{2Q_1} + \frac{\sqrt{-Q_2^2 Q_5^2 - Q_1^2 Q_6^2 + 2Q_1 Q_2 Q_5 Q_6 + 4Q_1 Q_2 Q_3 Q_4 - 4\delta(\delta Q_1 Q_2 - Q_5 Q_2 + Q_1 Q_6)}}{2Q_1 Q_2} > 0. \tag{47}$$

The study of the instability zones that correspond to conditions (45) and (47) is carried out in the next section. Before starting this study, however, it is important to note the following. The instability conditions (45) and (47) determine the first (main) instability zone of the parametric resonance. By analogy with the Mathieu's equation, it is natural to assume that there are more zones of the instability, which should occur under the condition $\chi V = 2(\Omega/n + \mu\delta)$, $n = 1, 2, \ldots$. To find these zones, one should modify the form of solution (26). The idea for such a modification should be taken from Ref. [21], where the same approach is used in the analysis of the higher order zones of the parametric resonance in the Mathieu's equation.

6. The instability zone

In this section, the instability zone is studied numerically. The study is performed using the following set of the system parameters:

$$\begin{aligned} &\rho = 7849\,\mathrm{kg}, \quad A_{cs} = 7.687 \times 10^{-3}\,\mathrm{m}^2, \\ &I = 3.055 \times 10^{-5}\,\mathrm{m}^4, \quad E = 2 \times 10^{11}\,\mathrm{N/m}^2, \\ &k_f = 10^8\,\mathrm{N/m}^2, \quad \mu = 0.3. \end{aligned} \tag{48}$$

First, the instability zone is studied in the case of the purely elastic foundation, e.g., with $v_f = 0$. In this case the instability zone is defined by the inequality (45). In Fig. 3 the centre of the instability zone ($\chi V - 2\Omega = 0$) is plotted in the plane "velocity-mass" for two periods of the inhomogeneity. These periods are chosen to represent the upper and the lower limits of the sleeper distance utilised in different types of the railway tracks. Fig. 3 shows that the larger the moving

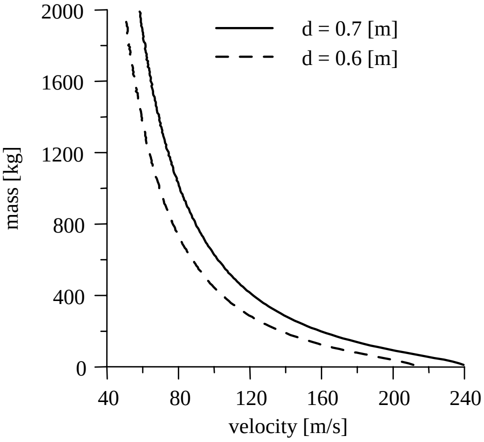

Fig. 3. Centre of the instability zone in the undamped case for two periods of the inhomogeneity: —, $d = 0.7$ m; – – –, $d = 0.6$ m.

mass and/or the smaller the period of the inhomogeneity, the smaller is the velocity at which the instability occurs.

In accordance with inequality (45), the boundaries of the instability zone are given by the equations

$$\chi V - 2\Omega = \pm 2\mu\sqrt{\frac{Q_3 Q_4}{Q_1 Q_2}}. \tag{49}$$

Because of the small parameter μ, the deviation of these boundaries from the centre of the zone is small and can be found in the following manner. Representing the velocity that corresponds to the boundary of the zone as $V = V_0 + \mu\tilde{V}$ with V_0 the velocity corresponding to the centre of the zone and $\mu\tilde{V}$ the small deviation of the velocity, Eq. (49) are rewritten as

$$(V_0 + \mu\tilde{V})\chi - 2\left(\Omega(V_0 + \mu\tilde{V}) \pm \mu\sqrt{\frac{Q_3(V_0 + \mu\tilde{V})Q_4(V_0 + \mu\tilde{V})}{Q_1(V_0 + \mu\tilde{V})Q_2(V_0 + \mu\tilde{V})}}\right) = 0. \tag{50}$$

Since $\mu\tilde{V}$ is assumed to be small, the function $\Omega(V_0 + \mu\tilde{V})$ can be expanded using the Taylor's series as

$$\Omega(V_0 + \mu\tilde{V}) = \Omega(V_0) + \left.\frac{\partial\Omega}{\partial V}\right|_{V_0} \cdot \tilde{V}. \tag{51}$$

Substituting expansion (51) into Eq. (50), taking into account that $\chi V_0 - 2\Omega(V_0) = 0$ (since V_0 corresponds to the centre of the zone) and collecting the terms of the order μ, the following expression for $\tilde{V}$ is obtained:

$$\tilde{V} = \mu\sqrt{\frac{Q_3(V_0)Q_4(V_0)}{Q_1(V_0)Q_2(V_0)}} \Bigg/ \left(\chi - 2\left.\frac{\partial\Omega}{\partial V}\right|_{V_0}\right). \tag{52}$$

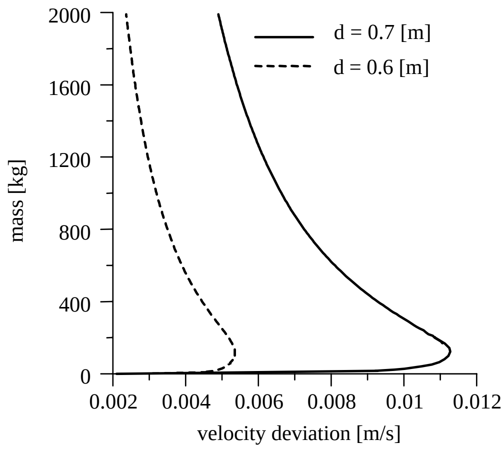

Fig. 4. Deviation of the boundaries of the instability zone from its centre: —, $d = 0.7$ m; – – –, $d = 0.6$ m.

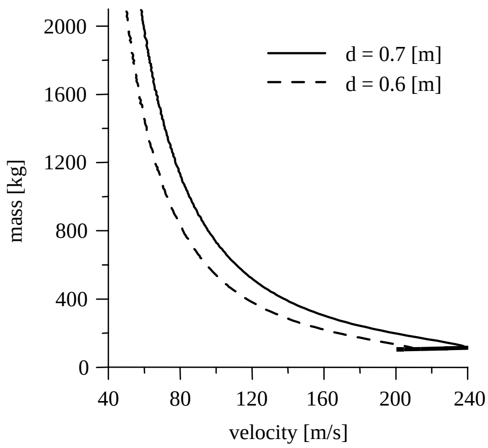

Fig. 5. Effect of the viscosity on the centre of the instability zone: —, $d = 0.7$ m; – – –, $d = 0.6$ m.

The deviation $\mu\tilde{V}$ of the velocity from the centre of the zone is presented in Fig. 4 as a function of the mass. This figure shows that the instability zone is very narrow, which makes it relatively easy to avoid the parametric instability in practice.

Consider the effect of the viscosity in the foundation on the instability zone. The centre of the zone is shown in Fig. 5 for $v_f = 100\,\text{Ns/m}^2$ and two periods of the inhomogeneity. From this figure, it can be seen that, in contrast to the undamped case, the instability does not arise if the mass is smaller than a critical value that is depicted with the help of the bold (almost horizontal) segment. Thus, analogous to the parametric resonance that is described by the Mathieu's equation, the effect of the viscosity leads to the shifting of the instability zone in the space of the system parameters.

Besides shifting the zone, the viscosity of the foundation makes the zone shrink in the velocity direction. However, for the chosen magnitude of the viscosity, this shrinkage is negligible.

7. Conclusions

In this paper, the stability of vibrations of a mass that moves uniformly along an Euler–Bernoulli beam on a periodically inhomogeneous foundation has been studied. It has been shown that these vibrations can become unstable due to the parametric resonance, which caused by the periodic variation of the foundation stiffness under the moving mass.

The first instability zone has been studied analytically by a perturbation method with the assumption that the variation of the foundation stiffness is small in comparison to the mean value of this stiffness. It has been found that the centre of the instability zone is defined by the condition that the doubled frequency of the mass vibrations on the homogeneous beam is close to the frequency of the stiffness variation under the moving mass. This condition is fully analogous to the condition of the parametric resonance in a system that is described by the Mathieu's equation.

It has been shown that the position of the instability zone in the system parameter depends strongly on the magnitude of the moving mass and the period of the inhomogeneity. The larger this period and/or the smaller the mass, the higher the velocity is at which the instability occurs. It is important to underline that, in principle, parametric instability can occur at any non-zero velocity of the mass. This is in contrast to the instability of a moving vehicle on a homogeneous guideway, which can occur only if the velocity exceeds the minimum phase velocity of waves in the guideway.

It has been found that the instability zone is very narrow with respect to the velocity of the mass. This is a natural consequence of the assumption that the inhomogeneity is weak.

The effect of the viscosity of the foundation has been studied. It has been found that this effect mainly leads to the shifting of the instability zone in the parameter space. This is also in perfect correspondence with the effect of the viscosity on the classical parametric resonance.

In conclusion, it is worth noting that the model employed in this paper cannot be considered as being able to describe the realistic train–track interaction. However, the main conclusion is quite general. It can be formulated as follows. If the inhomogeneity of a guideway is weak and periodic, then the parametric instability of a uniformly moving vehicle can occur. This instability should be expected when the frequency of variation of the guideway parameters under the moving vehicle is close to the doubled natural frequency of the vehicle as it moves uniformly along the guideway.

Acknowledgements

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Appendix A

In this Appendix, the constants are defined that are employed in expressions (13), (18), (19) and (20).

The constants from expression (13):

$$C_{A1}^{+} = -\frac{A(k_3^A - k_2^A)(k_4^A - k_2^A)}{(k_1^A + k_2^A - k_3^A - k_4^A)(k_1^A - k_2^A)}, \quad C_{A1}^{-} = \frac{A(k_4^A - k_1^A)(k_4^A - k_2^A)}{(k_1^A + k_2^A - k_3^A - k_4^A)(k_3^A - k_4^A)},$$

$$C_{A2}^{+} = \frac{A(k_3^A - k_1^A)(k_4^A - k_1^A)}{(k_1^A + k_2^A - k_3^A - k_4^A)(k_1^A - k_2^A)}, \quad C_{A2}^{-} = -\frac{A\left(k_3^A - k_1^A\right)\left(k_3^A - k_2^A\right)}{(k_1^A + k_2^A - k_3^A - k_4^A)(k_3^A - k_4^A)},$$

$$C_{B1}^{+} = -\frac{B(k_3^B - k_2^B)(k_4^B - k_2^B)}{(k_1^B + k_2^B - k_3^B - k_4^B)(k_1^B - k_2^B)}, \quad C_{B1}^{-} = \frac{B(k_4^B - k_1^B)(k_4^B - k_2^B)}{(k_1^B + k_2^B - k_3^B - k_4^B)(k_3^B - k_4^B)},$$

$$C_{B2}^{+} = \frac{B(k_3^B - k_1^B)(k_4^B - k_1^B)}{(k_1^B + k_2^B - k_3^B - k_4^B)(k_1^B - k_2^B)}, \quad C_{B2}^{-} = -\frac{B(k_3^B - k_1^B)(k_3^B - k_2^B)}{(k_1^B + k_2^B - k_3^B - k_4^B)(k_3^B - k_4^B)}.$$

The constants from expressions (18) and (19):

$$C_{11}^{+} = \frac{-k_f C_{A1}^{+}}{2EI\chi(\chi + 2k_1^A)((k_1^A)^2 + (k_1^A + \chi)^2)}, \quad C_{12}^{+} = \frac{-k_f C_{A1}^{+}}{2EI\chi(\chi - 2k_1^A)((k_1^A)^2 + (k_1^A - \chi)^2)},$$

$$C_{21}^{+} = \frac{-k_f C_{A2}^{+}}{2EI\chi(\chi + 2k_2^A)((k_2^A)^2 + (k_2^A + \chi)^2)}, \quad C_{22}^{+} = \frac{-k_f C_{A2}^{+}}{2EI\chi(\chi - 2k_2^A)((k_2^A)^2 + (k_2^A - \chi)^2)},$$

$$C_{31}^{+} = \frac{-k_f C_{B1}^{+}}{2EI\chi(\chi + 2k_1^B)((k_1^B)^2 + (k_1^B + \chi)^2)}, \quad C_{32}^{+} = \frac{-k_f C_{B1}^{+}}{2EI\chi(\chi - 2k_1^B)((k_1^B)^2 + (k_1^B - \chi)^2)},$$

$$C_{41}^{+} = \frac{-k_f C_{B2}^{+}}{2EI\chi(\chi + 2k_2^B)((k_2^B)^2 + (k_2^B + \chi)^2)}, \quad C_{42}^{+} = \frac{-k_f C_{B2}^{+}}{2EI\chi(\chi - 2k_2^B)((k_2^B)^2 + (k_2^B - \chi)^2)},$$

$$C_{11}^{-} = \frac{-k_f C_{A1}^{-}}{2EI\chi(\chi + 2k_3^A)((k_3^A)^2 + (k_3^A + \chi)^2)}, \quad C_{12}^{-} = \frac{-k_f C_{A1}^{-}}{2EI\chi(\chi - 2k_3^A)((k_3^A)^2 + (k_3^A - \chi)^2)},$$

$$C_{21}^{-} = \frac{-k_f C_{A2}^{-}}{2EI\chi(\chi + 2k_4^A)((k_4^A)^2 + (k_4^A + \chi)^2)}, \quad C_{22}^{-} = \frac{-k_f C_{A2}^{-}}{2EI\chi(\chi - 2k_4^A)((k_4^A)^2 + (k_4^A - \chi)^2)},$$

$$C_{31}^{-} = \frac{-k_f C_{B1}^{-}}{2EI\chi(\chi + 2k_3^B)((k_3^B)^2 + (k_3^B + \chi)^2)}, \quad C_{32}^{-} = \frac{-k_f C_{B1}^{-}}{2EI\chi(\chi - 2k_3^B)((k_3^B)^2 + (k_3^B - \chi)^2)},$$

$$C_{41}^{-} = \frac{-k_f C_{B2}^{-}}{2EI\chi(\chi + 2k_4^B)((k_4^B)^2 + (k_4^B + \chi)^2)}, \quad C_{42}^{-} = \frac{-k_f C_{B2}^{-}}{2EI\chi(\chi - 2k_4^B)((k_4^B)^2 + (k_4^B - \chi)^2)}.$$

The constants from expression (20):

$$D_{11} = i((k_1^A + \chi)C_{11}^{+} + (k_2^A + \chi)C_{21}^{+} - (k_3^A + \chi)C_{11}^{-} - (k_4^A + \chi)C_{21}^{-}),$$

$$D_{12} = i((k_1^A - \chi)C_{12}^{+} + (k_2^A - \chi)C_{22}^{+} - (k_3^A - \chi)C_{12}^{-} - (k_4^A - \chi)C_{22}^{-}),$$

$$D_{13} = i((k_1^B + \chi)C_{31}^{+} + (k_2^B + \chi)C_{41}^{+} + (k_3^B + \chi)C_{31}^{-} + (k_4^B + \chi)C_{41}^{-}),$$

$$D_{14} = i((k_1^B - \chi)C_{32}^{+} + (k_2^B - \chi)C_{42}^{+} + (k_3^B - \chi)C_{32}^{-} - (k_4^B + \chi)C_{42}^{-}),$$

$$\begin{aligned} D_{21} &= -((k_1^A+\chi)^2 C_{11}^+ + (k_2^A+\chi)^2 C_{21}^+ - (k_3^A+\chi)^2 C_{11}^- - (k_4^A+\chi)^2 C_{21}^-),\\ D_{22} &= -((k_1^A-\chi)^2 C_{12}^+ + (k_2^A-\chi)^2 C_{22}^+ - (k_3^A-\chi)^2 C_{12}^- - (k_4^A-\chi)^2 C_{22}^-),\\ D_{23} &= -((k_1^B+\chi)^2 C_{31}^+ + (k_2^B+\chi)^2 C_{41}^+ + (k_3^B+\chi)^2 C_{31}^- + (k_4^B+\chi)^2 C_{41}^-),\\ D_{24} &= -((k_1^B-\chi)^2 C_{32}^+ + (k_2^B-\chi)^2 C_{42}^+ + (k_3^B-\chi)^2 C_{32}^- - (k_4^B+\chi)^2 C_{42}^-), \end{aligned}$$

$$D_{31} = -(C_{11}^+ + C_{21}^+),\quad D_{32} = -(C_{12}^+ + C_{22}^+),\quad D_{33} = -(C_{31}^+ + C_{41}^+),\quad D_{34} = -(C_{32}^+ + C_{42}^+),$$

$$\begin{aligned} D_{41} &= i((k_1^A+\chi)^3 C_{11}^+ + (k_2^A+\chi)^3 C_{21}^+ - (k_3^A+\chi)^3 C_{11}^- - (k_4^A+\chi)^3 C_{21}^-),\\ D_{42} &= i((k_1^A-\chi)^3 C_{12}^+ + (k_2^A-\chi)^3 C_{22}^+ - (k_3^A-\chi)^3 C_{12}^- - (k_4^A-\chi)^3 C_{22}^-),\\ D_{43} &= i((k_1^B+\chi)^3 C_{31}^+ + (k_2^B+\chi)^3 C_{41}^+ + (k_3^B+\chi)^3 C_{31}^- + (k_4^B+\chi)^3 C_{41}^-),\\ D_{44} &= i((k_1^B-\chi)^3 C_{32}^+ + (k_2^B-\chi)^3 C_{42}^+ + (k_3^B-\chi)^3 C_{32}^- - (k_4^B+\chi)^3 C_{42}^-). \end{aligned}$$

Appendix B

The system of equations that is obtained by substitution of expressions (26) into the system of Eq. (2) followed by collection of terms of the order μ^0 reads

For $x > Vt$,

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}t(\Omega+\mu\delta)} \sum_{j=1}^{2} C_{Aj}^+(\mu x, \mu t)\mathrm{e}^{\mathrm{i}k_j^A(Vt-x)}(-\rho A_{cs}(\Omega + k_j^A V)^2 + EI(k_j^A)^4 + k_f)\\ &\quad + \mathrm{e}^{-\mathrm{i}t(\Omega+\mu\delta)} \sum_{j=1}^{2} C_{Bj}^+(\mu x, \mu t)\mathrm{e}^{\mathrm{i}k_j^B(Vt-x)}(-\rho A_{cs}(-\Omega + k_j^B V)^2 + EI(k_j^B)^4 + k_f) = 0. \end{aligned} \tag{B.1}$$

For $x < Vt$,

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}t(\Omega+\mu\delta)} \sum_{j=1}^{2} C_{Aj}^-(\mu x, \mu t)\mathrm{e}^{\mathrm{i}k_{j+2}^A(Vt-x)}(-\rho A_{cs}(\Omega + k_{j+2}^A V)^2 + EI(k_{j+2}^A)^4 + k_f)\\ &\quad + \mathrm{e}^{-\mathrm{i}t(\Omega+\mu\delta)} \sum_{j=1}^{2} C_{Bj}^-(\mu x, \mu t)\mathrm{e}^{\mathrm{i}k_{j+2}^B(Vt-x)}(-\rho A_{cs}(-\Omega + k_{j+2}^A V)^2 + EI(k_{j+2}^B)^4 + k_f) = 0. \end{aligned} \tag{B.2}$$

For $x = Vt$,

$$\begin{aligned} &\sum_{j=1}^{2} C_{Aj}^+(\mu Vt, \mu t)\mathrm{e}^{\mathrm{i}t(\Omega+\mu\delta)} + \sum_{j=1}^{2} C_{Bj}^+(\mu Vt, \mu t)e^{-\mathrm{i}t(\Omega+\mu\delta)}\\ &\quad = \sum_{j=1}^{2} C_{Aj}^-(\mu Vt, \mu t)\mathrm{e}^{\mathrm{i}t(\Omega+\mu\delta)} + \sum_{j=1}^{2} C_{Bj}^-(\mu Vt, \mu t)\mathrm{e}^{-\mathrm{i}t(\Omega+\mu\delta)}. \end{aligned} \tag{B.3}$$

$$\sum_{j=1}^{2} k_j^A C_{Aj}^{+}(\mu Vt, \mu t)e^{it(\Omega+\mu\delta)} + \sum_{j=1}^{2} k_j^B C_{Bj}^{+}(\mu Vt, \mu t)e^{-it(\Omega+\mu\delta)}$$
$$= \sum_{j=1}^{2} k_{j+2}^A C_{Aj}^{-}(\mu Vt, \mu t)e^{it(\Omega+\mu\delta)} + \sum_{j=1}^{2} k_{j+2}^B C_{Bj}^{-}(\mu Vt, \mu t)e^{-it(\Omega+\mu\delta)}, \quad \text{(B.4)}$$

$$\sum_{j=1}^{2} (k_j^A)^2 C_{Aj}^{+}(\mu Vt, \mu t)e^{it(\Omega+\mu\delta)} + \sum_{j=1}^{2} (k_j^B)^2 C_{Bj}^{+}(\mu Vt, \mu t)e^{-it(\Omega+\mu\delta)}$$
$$= \sum_{j=1}^{2} (k_{j+2}^A)^2 C_{Aj}^{-}(\mu Vt, \mu t)e^{it(\Omega+\mu\delta)} + \sum_{j=1}^{2} (k_{j+2}^B)^2 C_{Bj}^{-}(\mu Vt, \mu t)e^{-it(\Omega+\mu\delta)}, \quad \text{(B.5)}$$

$$\sum_{j=1}^{2} C_{Aj}^{+}(\mu Vt, \mu t)e^{it(\Omega+\mu\delta)} + \sum_{j=1}^{2} C_{Bj}^{+}(\mu Vt, \mu t)e^{-it(\Omega+\mu\delta)} = A(\mu t)e^{it(\Omega+\mu\delta)} + B(\mu t)e^{-it(\Omega+\mu\delta)}, \quad \text{(B.6)}$$

$$EI\left(\sum_{j=1}^{2} i(k_j^A)^3 C_{Aj}^{+}(\mu Vt, \mu t)e^{it(\Omega+\mu\delta)} + \sum_{j=1}^{2} i(k_j^B)^3 C_{Bj}^{+}(\mu Vt, \mu t)e^{-it(\Omega+\mu\delta)}\right.$$
$$\left. - \sum_{j=1}^{2} i(k_{j+2}^A)^3 C_{Aj}^{-}(\mu Vt, \mu t)e^{it(\Omega+\mu\delta)} - \sum_{j=1}^{2} i(k_{j+2}^B)^3 C_{Bj}^{-}(\mu Vt, \mu t)e^{-it(\Omega+\mu\delta)}\right)$$
$$= m\Omega^2(A(\mu t)e^{it(\Omega+\mu\delta)} + B(\mu t)e^{-it(\Omega+\mu\delta)}). \quad \text{(B.7)}$$

Obviously, Eqs. (B.1) and (B.2) are satisfied automatically, since the wavenumbers $k_{1,2,3,4}^{A,B}$ are the roots of the dispersion Eq. (8). The Eqs. (B.3)–(B.7) can be subdivided into two systems of equations, one containing the terms proportional to $e^{it(\Omega+\mu\delta)}$ and the other one with the terms proportional to $e^{-it(\Omega+\mu\delta)}$. The natural frequency Ω is the eigenvalue of the determinants of both these systems. Therefore, Eqs. (B.3)–(B.7), as well as Eqs. (B.1) and (B.2) are satisfied independently of the choice of the amplitudes $C_{Aj}^{\pm}$ and $C_{Bj}^{\pm}$.

Appendix C

In this appendix, the constants are defined that are employed in expressions (38) and (39).

$$\tilde{C}_{11}^{+} = \frac{-k_f C_{A1}^{+}(\mu x, \mu t)}{2EI\chi(\chi + 2k_1^A)((k_1^A)^2 + (k_1^A + \chi)^2)}, \quad \tilde{C}_{12}^{+} = \frac{-k_f C_{A2}^{+}(\mu x, \mu t)}{2EI\chi(\chi + 2k_2^A)((k_2^A)^2 + (k_2^A + \chi)^2)},$$

$$\tilde{C}_{21}^{+} = \frac{-k_f C_{B1}^{+}(\mu x, \mu t)}{2EI\chi(\chi - 2k_1^B)((k_1^B)^2 + (k_1^B - \chi)^2)}, \quad \tilde{C}_{22}^{+} = \frac{-k_f C_{B2}^{+}(\mu x, \mu t)}{2EI\chi(\chi - 2k_2^B)((k_2^B)^2 + (k_2^B - \chi)^2)},$$

$$\tilde{C}_{11}^{-} = \frac{-k_f C_{A1}^{-}(\mu x, \mu t)}{2EI\chi(\chi + 2k_3^A)((k_3^A)^2 + (k_3^A + \chi)^2)}, \quad \tilde{C}_{12}^{-} = \frac{-k_f C_{A2}^{-}(\mu x, \mu t)}{2EI\chi(\chi + 2k_4^A)((k_4^A)^2 + (k_4^A + \chi)^2)},$$

$$\tilde{C}_{21}^{-} = \frac{-k_f C_{B1}^{-}(\mu x, \mu t)}{2EI\chi(\chi - 2k_3^B)((k_3^B)^2 + (k_3^B - \chi)^2)}, \quad \tilde{C}_{22}^{-} = \frac{-k_f C_{B2}^{-}(\mu x, \mu t)}{2EI\chi(\chi - 2k_4^B)((k_4^B)^2 + (k_4^B - \chi)^2)}.$$

Appendix D

In this appendix, the relations are presented that are obtained by substitution of expressions (42) into Eqs. (B.3)–(B.6). Further, expressions are given for the constants $D_j,\ j = 1, \dots, 6$ that are employed in the equations.

Relations obtained by substituting (42) into Eqs. (B.3)–(B.6):

$$q_1^A - p_1^A V = q_2^A - p_2^A V = q_1^B - p_1^B V = q_2^B - p_2^B V = q_3^A - p_3^A V$$
$$= q_4^A - p_4^A V = q_3^B - p_3^B V = q_4^B - p_4^B V = s,$$

$$C_{A10}^{+} = -\frac{A_0(k_3^A - k_2^A)(k_4^A - k_2^A)}{(k_1^A + k_2^A - k_3^A - k_4^A)(k_1^A - k_2^A)}, \quad C_{A10}^{-} = \frac{A_0(k_4^A - k_1^A)(k_4^A - k_2^A)}{(k_1^A + k_2^A - k_3^A - k_4^A)(k_3^A - k_4^A)},$$

$$C_{A20}^{+} = \frac{A_0(k_3^A - k_1^A)(k_4^A - k_1^A)}{(k_1^A + k_2^A - k_3^A - k_4^A)(k_1^A - k_2^A)}, \quad C_{A20}^{-} = -\frac{A_0(k_3^A - k_1^A)(k_3^A - k_2^A)}{(k_1^A + k_2^A - k_3^A - k_4^A)(k_3^A - k_4^A)},$$

$$C_{B10}^{+} = -\frac{B_0(k_3^B - k_2^B)(k_4^B - k_2^B)}{(k_1^B + k_2^B - k_3^B - k_4^B)(k_1^B - k_2^B)}, \quad C_{B10}^{-} = \frac{B_0(k_4^B - k_1^B)(k_4^B - k_2^B)}{(k_1^B + k_2^B - k_3^B - k_4^B)(k_3^B - k_4^B)},$$

$$C_{B20}^{+} = \frac{B_0(k_3^B - k_1^B)(k_4^B - k_1^B)}{(k_1^B + k_2^B - k_3^B - k_4^B)(k_1^B - k_2^B)}, \quad C_{B20}^{-} = -\frac{B_0(k_3^B - k_1^B)(k_3^B - k_2^B)}{(k_1^B + k_2^B - k_3^B - k_4^B)(k_3^B - k_4^B)}. \tag{D.1}$$

The constants from expression (43):

$$\begin{aligned}
Q_1 = &-2m\Omega(k_1^A + k_2^A - k_3^A - k_4^A) \\
&+ \frac{3iEI\rho A_{cs}}{(k_1^A - k_2^A)}\left(-\frac{(k_1^A V + \Omega)(k_1^A)^2(k_3^A - k_2^A)(k_4^A - k_2^A)}{(\rho A_{cs} V(k_1^A V + \Omega) + 2EI(k_1^A)^3)} + \frac{(k_2^A V + \Omega)(k_2^A)^2(k_3^A - k_1^A)(k_4^A - k_1^A)}{(\rho A_{cs} V(k_2^A V + \Omega) + 2EI(k_2^A)^3)}\right) \\
&- \frac{3iEI\rho A_{cs}}{(k_3^A - k_4^A)}\left(\frac{(k_3^A V + \Omega)(k_3^A)^2(k_4^A - k_1^A)(k_4^A - k_2^A)}{(\rho A_{cs} V(k_3^A V + \Omega) + 2EI(k_3^A)^3)} - \frac{(k_4^A V + \Omega)(k_4^A)^2(k_3^A - k_1^A)(k_3^A - k_2^A)}{(\rho A_{cs} V(k_4^A V + \Omega) + 2EI(k_4^A)^3)}\right),
\end{aligned}$$

$$\begin{aligned}
Q_2 = &\,2m\Omega(k_1^B + k_2^B - k_3^B - k_4^B) \\
&+ \frac{3iEI\rho A_{cs}}{(k_1^B - k_2^B)}\left(-\frac{(k_1^B V - \Omega)(k_1^B)^2(k_3^B - k_2^B)(k_4^B - k_2^B)}{(\rho A_{cs} V(k_1^B V - \Omega) + 2EI(k_1^B)^3)} + \frac{(k_2^B V - \Omega)(k_2^B)^2(k_3^B - k_1^B)(k_4^B - k_1^B)}{(\rho A_{cs} V(k_2^B V - \Omega) + 2EI(k_2^B)^3)}\right) \\
&- \frac{3iEI\rho A_{cs}}{(k_3^B - k_4^B)}\left(\frac{(k_3^B V - \Omega)(k_3^A)^2(k_4^B - k_1^B)(k_4^B - k_2^B)}{(\rho A_{cs} V(k_3^B V - \Omega) + 2EI(k_3^B)^3)} - \frac{(k_4^B V - \Omega)(k_4^B)^2(k_3^B - k_1^B)(k_3^B - k_2^B)}{(\rho A_{cs} V(k_4^B V - \Omega) + 2EI(k_4^B)^3)}\right),
\end{aligned}$$

$$Q_3 = -\frac{\mathrm{i}k_f}{(k_1^A - k_2^A)}\left(-\frac{(k_1^A+\chi)^3(k_3^A-k_2^A)(k_4^A-k_2^A)}{2\chi(\chi+2k_1^A)((k_1^A)^2+(k_1^A+\chi)^2)} + \frac{(k_2^A+\chi)^3(k_3^A-k_1^A)(k_4^A-k_1^A)}{2\chi(\chi+2k_2^A)((k_2^A)^2+(k_2^A+\chi)^2)}\right)$$
$$+\frac{\mathrm{i}k_f}{(k_3^A - k_4^A)}\left(\frac{(k_3^A+\chi)^3(k_4^A-k_1^A)(k_4^A-k_2^A)}{2\chi(\chi+2k_3^A)((k_3^A)^2+(k_3^A+\chi)^2)} - \frac{(k_4^A+\chi)^3(k_3^A-k_1^A)(k_3^A-k_2^A)}{2\chi(\chi+2k_4^A)((k_4^A)^2+(k_4^A+\chi)^2)}\right),$$

$$Q_4 = \frac{\mathrm{i}k_f}{(k_1^B - k_2^B)}\left(-\frac{(k_1^B-\chi)^3(k_3^B-k_2^B)(k_4^B-k_2^B)}{2\chi(\chi-2k_1^B)((k_1^B)^2+(k_1^B-\chi)^2)} + \frac{(k_2^B-\chi)^3(k_3^B-k_1^B)(k_4^B-k_1^B)}{2\chi(\chi-2k_2^B)((k_2^B)^2+(k_2^B-\chi)^2)}\right)$$
$$-\frac{\mathrm{i}k_f}{(k_3^B - k_4^B)}\left(\frac{(k_3^B-\chi)^3(k_4^B-k_1^B)(k_4^B-k_2^B)}{2\chi(\chi-2k_3^B)((k_3^B)^2+(k_3^B-\chi)^2)} - \frac{(k_4^B-\chi)^3(k_3^B-k_1^B)(k_3^B-k_2^B)}{2\chi(\chi-2k_4^B)((k_4^B)^2+(k_4^B-\chi)^2)}\right).$$

The constants from expression (46):

$$Q_5 = -\frac{3v_f\rho A_{cs}V}{(k_1^A - k_2^A)}\left(-\frac{(k_1^AV+\Omega)^2(k_3^A-k_2^A)(k_4^A-k_2^A)}{2k_1^A(\rho A_{cs}V(k_1^AV+\Omega)+2EI(k_1^A)^3)} + \frac{(k_2^AV+\Omega)^2(k_3^A-k_1^A)(k_4^A-k_1^A)}{2k_2^A(\rho A_{cs}V(k_2^AV+\Omega)+2EI(k_2^A)^3)}\right)$$
$$+\frac{3v_f\rho A_{cs}V}{(k_3^A - k_4^A)}\left(\frac{(k_3^AV+\Omega)^2(k_4^A-k_1^A)(k_4^A-k_2^A)}{2k_3^A(\rho A_{cs}V(k_3^AV+\Omega)+2EI(k_3^A)^3)} - \frac{(k_4^AV+\Omega)^2(k_3^A-k_1^A)(k_3^A-k_2^A)}{2k_4^A(\rho A_{cs}V(k_4^AV+\Omega)+2EI(k_4^A)^3)}\right)$$

$$Q_6 = \frac{3v_f\rho A_{cs}V}{(k_1^B - k_2^B)}\left(-\frac{(k_1^BV-\Omega)^2(k_3^B-k_2^B)(k_4^B-k_2^B)}{2k_1^B(\rho A_{cs}V(k_1^BV-\Omega)+2EI(k_1^B)^3)} + \frac{(k_2^BV-\Omega)^2(k_3^B-k_1^B)(k_4^B-k_1^B)}{2k_2^B(\rho A_{cs}V(k_2^BV-\Omega)+2EI(k_2^B)^3)}\right)$$
$$-\frac{3v_f\rho A_{cs}V}{(k_3^B - k_4^B)}\left(\frac{(k_3^BV-\Omega)^2(k_4^B-k_1^B)(k_4^B-k_2^B)}{2k_3^B(\rho A_{cs}V(k_3^BV-\Omega)+2EI(k_3^B)^3)} - \frac{(k_4^BV-\Omega)^2(k_3^B-k_1^B)(k_3^B-k_2^B)}{2k_4^B(\rho A_{cs}V(k_4^BV-\Omega)+2EI(k_4^B)^3)}\right),$$

Appendix E. Nomenclature

A_{cs}	the cross-sectional area of the beam
A, B, C, D	the wave amplitudes
d	the period of the inhomogeneity
E	Young's modulus
i	$= \sqrt{-1}$
I	the moment of inertia of the beam's cross-section
k_f	the mean stiffness of the foundation
k_n	the roots of the dispersion relation which possess a positive imaginary part
$k_n^{A,B}$	the wavenumbers
m	the mass
Q_j	constants
t	time
V	the velocity of motion
w	the vertical deflection of the beam relative to its equilibrium position

w_0	the vertical deflection of the mass relative to its equilibrium position
x	the horizontal coordinate
χ	the wave number of inhomogeneity
δ	the mistuning
$\delta(\ldots)$	the Dirac's delta function
μ	the dimensionless small parameter
v_f	the viscosity of foundation
ρ	the mass density of the beam material
Ω	the radial frequency

References

[1] G.G. Denisov, E.K. Kugusheva, V.V. Novikov, On the problem of the stability of one-dimensional unbounded elastic systems, Journal of Applied Mathematics and Mechanics 49 (4) (1985) 533–537.

[2] R. Bogacz, S. Novakowski, K. Popp, On the stability of a Timoshenko beam on an elastic foundation under a moving spring-mass system, Acta Mechanica 61 (1986) 117–127.

[3] A.V. Metrikine, Unstable lateral oscillations of an object moving uniformly along elastic guide as a result of anomalous Doppler effect, Acoustical Physics 40 (1) (1994) 85–89.

[4] A.V. Metrikine, H.A. Dieterman, Instability of vibrations of a mass moving uniformly along an axially compressed beam on a visco-elastic foundation, Journal of Sound and Vibrations 201 (5) (1997) 567–576.

[5] A.R.M. Wolfert, H.A. Dieterman, A.V. Metrikine, Stability of vibrations of two oscillators moving uniformly along a beam on a viscoelastic foundation, Journal of Sound and Vibrations 211 (5) (1998) 829–842.

[6] A.V. Metrikine, K. Popp, Instability of vibrations of an oscillator moving along a beam on an elastic half-space, European Journal of Mechanics A, Solids 18 (2) (1999) 331–349.

[7] D.Y. Zheng, F.T.K. Au, Y.K. Cheung, Vibration of vehicle on compressed rail on viscoelastic foundation, Journal of Engineering Mechanics 126 (11) (2000) 1141–1147.

[8] A.V. Metrikine, S.N. Verichev, Instability of vibration of a moving oscillator on a flexibly supported Timoshenko beam, Archive of Applied Mechanics 71 (9) (2001) 613–624.

[9] A.D. Kerr, The continuously supported rail subjected to an axial force and a moving load, International Journal of Mechanical Sciences 14 (1972) 71–78.

[10] J.J. Labra, An axially stressed railroad track on an elastic continuum subjected to a moving load, Acta Mechanica 22 (1975) 113–129.

[11] D.J. Mead, Vibration response and wave propagation in periodic structures, Transactions of ASME, Journal of Engineering for Industry Series B 93 (1971) 783–792.

[12] L. Jezequel, Analysis of critical speeds of a moving load on an infinite periodically supported beam, Journal of Sound and Vibration 73 (4) (1980) 606–610.

[13] R. Bogacz, T. Krzyzynski, K. Popp, On dynamics of systems modelling continuous and periodic guideways, Archive of Applied Mechanics 45 (5) (1993) 575–593.

[14] A.I. Vesnitskii, A.V. Metrikine, Transient radiation in a periodically inhomogeneous elastic guide, Mechanics of Solids 28 (6) (1993) 158–162.

[15] P.M. Belotserkovskiy, On the oscillations of infinite periodic beams subjected to a moving concentrated force, Journal of Sound and Vibration 193 (3) (1996) 705–712.

[16] P.M. Belotserkovskiy, Forced oscillations and resonance of infinite periodic strings, Journal of Sound and Vibration 204 (1) (1997) 41–57.

[17] A.V. Metrikine, K. Popp, Vibration of a periodically supported beam on an elastic half-space, European Journal of Mechanics A, Solids 18 (4) (1999) 679–701.

[18] Y.I. Chung, J. Genin, Stability of a vehicle on a multispan simply supported guideway, Transactions of ASME, Journal of Dynamic Systems, Measurement and Control 100 (4) (1978) 326–332.

[19] A.I. Vesnitskii, A.V. Metrikine, Parametric instability in the oscillations of a body moving uniformly in a periodically inhomogeneous elastic system, Journal of Applied Mechanics and Technical Physics 34 (2) (1993) 266–271.
[20] A.H. Nayfeh, Perturbation Methods, Wiley, New York, 2000.
[21] L.D. Landau, E.M. Lifshitz, Mechanics, Pergamon, Oxford, 1976.
[22] F.M. Arscott, Periodic Differential Equations, Pergamon, Oxford, 1964.